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Characteristics method for the formulation and computation of a free boundary cavitation problem

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Abstract

In this paper, a semidiscretized scheme based on characteristics method is analyzed when it is applied to a free boundary problem. The free boundary problem issued from a cavitation model in lubrication involves specific boundary conditions. The idea is to associate to the departure problem a sequence of variational inequalities depending on a discretization parameter. The theoretical convergence result from the solution of the semidiscretized problem to the solution of the continuous one is stated for a flux-imposed boundary condition. For this purpose, due to the specific boundary conditions, technical modifications of the classical version of characteristics are needed. Then, obstacle problem tools are applied to a set of variational inequalities with Neumann boundary conditions in order to obtain $L^\infty(\Omega)$ estimates which provide the convergence result. In this way, a result of existence of solution for the continuous problem and a theoretical justification for the usage of the proposed numerical methods are concluded. Finally, some numerical test examples are presented to illustrate the good performance of the method.

AMS classification: 65M; 35Q

Keywords: Lubrication; Cavitation; Free boundary problems; Finite elements; Duality methods

1. Introduction

Axially lubricated journal bearings are often present in industrial applications. The device consists of a circular cylinder (*the journal*) which rotates inside another one (*the bearing*). In order to avoid damage to the surfaces (by friction, heating, etc.), the gap between both cylinders is lubricated

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by means of grease or oil. The supply of the fluid can be made through a hole, an axial or a circumferential groove.

The mathematical model for the fluid pressure behavior inside the journal-bearing gap is mainly based on the Reynolds equation which represents an asymptotic bidimensional approximation of the real three-dimensional problem [4]. Moreover, an additional phenomenon has been experimentally tested: the presence of air bubbles in the lubricant (*cavitation*) due to the convergent–divergent geometry. The most extended mathematical models for cavitation are the Reynolds and Elrod–Adams ones (see [3]; for the comparison between different models). The model of Reynolds can be written in terms of a variational inequality and therefore the classical theoretical results and numerical methods can be applied. Nevertheless, the variational inequality formulation imposes the cavitation to happen in the part of an increasing gap. But, in some cases, such as journal bearings supplied with oil by a drip feed, the device may not receive enough lubricant to produce a full film in the part of decreasing gap. This kind of cavitation phenomenon is known as starvation. The model of Elrod–Adams introduces the saturation of the fluid at each point as an additional unknown and allows the starvation to take place. Existence and uniqueness have been studied by means of Schauder fixed point method in [2, 19]. Mathematical aspects of this problem are very close to the dam problem so that stationary numerical methods based upon upwind finite elements proposed in [1, 18, 17] for the dam problem have been used in [14, 2]. Recently, Boukrouche and Bayada in [9] established the existence of solution for the same kind of problem but with periodic boundary conditions. Numerical experimentations of various schemes based both upon stationary upwind methods and pseudo-instationary techniques have been conducted in [11].

This paper is devoted to a constructive approach by means of the characteristics method applied to a problem whose existence and uniqueness of solution has been obtained in the previous work [19]. Due to the ability of combining the characteristics method and finite elements approach, and the suppression of numerical oscillations by upwinding techniques, such an approach has been widely used for linear stationary and unstationary problems but few results appear in the literature for nonlinear problems. In this paper we will prove that a specific semidiscretization leads to a well-posed time-discretized problem and then we will obtain the theoretical convergence of this problem which justifies the numerical performance of the method.

In Section 2, the mathematical formulation is presented. It corresponds to the Elrod–Adams model for cavitation in the lubrication of a journal bearing with axial supply. The set of equations defines a nonlinear elliptic free boundary problem where the free boundary separates the fluid and cavitated regions.

In Section 3 the method of characteristics is proposed to obtain a numerical procedure to approximate the solution. The method has been introduced in [5] for convection–diffusion evolutive equations and adapted in [7] for the stationary case. The same authors have applied this adaptation of the method for a lubrication problem with periodic boundary condition in [8].

In Section 4 the discretized formulation of characteristics is related to a first kind elliptic variational inequality with Neumann boundary conditions in order to apply the classical obstacle problem tools which can be found in [16], for example, and conclude the existence and uniqueness of solution for this discretized formulation. Moreover, a regularity property and an $L^\infty(\Omega)$ estimate are obtained.

In Section 5 the previous estimate combined with some analytical computations justify the convergence of the approximated solution to the one of the initial lubrication problem.

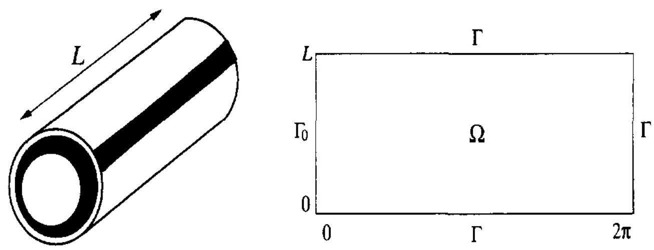

Fig. 1. Journal-bearing device and bidimensional domain.

In Section 6, some numerical tests are presented by using the Bermúdez–Moreno algorithm for solving the variational inequality equivalent to the semidiscretized problem.

Finally, some conclusions and further possible applications to other analogous test problems are proposed in Section 7.

2. The model problem

The study of the journal-bearing device with axial supply gives rise to a mathematical formulation in the bidimensional domain $\Omega=(0,2\pi)\times(0,L)$ where L is the length of the cylinders which is taken equal to one for the sake of simplicity. In this geometry the supply of lubricant is made through the left boundary Γ_0 which corresponds to $x=0$, see Fig. 1.

The lubricant is considered homogeneous and isoviscous. As in the previous work [19], it can be shown that, under certain conditions, the piezoviscous fluids whose viscosity obeys Barus' law can be reduced to isoviscous ones by means of the appropriate change of variable [12]. Moreover, the viscosity coefficient will be normalized to one for clearness.

Thus, the strong formulation of the initial problem remains as follows:

Find (q,θ) such that:

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial q}{\partial x}\right)+\frac{\partial}{\partial y}\left(h^3\frac{\partial q}{\partial y}\right)=\frac{\partial h}{\partial x},\quad q>0 \quad\text{and}\quad \theta=1 \text{ in } \Omega^+, \tag{2.1}$$

$$\frac{\partial}{\partial x}(\theta h)=0,\ q=0 \quad\text{and}\quad 0\leqslant\theta\leqslant 1 \text{ in } \Omega_0, \tag{2.2}$$

$$h^3\frac{\partial q}{\partial n}=(1-\theta)h\cos(\boldsymbol{n},\boldsymbol{\imath}),\ q=0 \quad\text{on } \Sigma, \tag{2.3}$$

$$\theta=\theta_0 \quad\text{on } \Gamma_0, \tag{2.4}$$

$$q=0 \quad\text{on } \Gamma, \tag{2.5}$$

where the given function h represents the gap between the two cylinders and it is given, for example, by

$$h=h(x)=(1+\varepsilon\cos(x)),\quad \forall x\in[0,2\pi]$$

with ε being the eccentricity parameter which is greater than zero and strictly less than one, $\boldsymbol{n}$ being the normal vector to the free boundary Σ and $(\boldsymbol{n},\boldsymbol{i})$ being the angle between $\boldsymbol{n}$ and $\boldsymbol{i}$ ($\boldsymbol{i}$ is the unitary vector in the x-direction). The unknown functions q and θ represent the reduced pressure of the lubricant and the fluid concentration (which is equal to one in the fluid region and between zero and one in the cavitation region). The previously appearing sets are defined by

$$
\begin{aligned}
\Omega\ &= (0,2\pi)\times(0,1),\\
\Omega^+ &= \{(x,y)\in\Omega/q(x,y)>0\},\\
\Omega_0 &= \{(x,y)\in\Omega/q(x,y)=0\},\\
\Sigma\ &= \partial\Omega^+\cap\Omega,\\
\Gamma_0 &= \{(x,y)\in\partial\Omega/x=0\},\\
\Gamma\ &= \partial\Omega-\Gamma_0.
\end{aligned}
\tag{2.6}
$$

Eq. (2.1) represents the Reynolds equation in the active region full of fluid noted as Ω^+. Eq. (2.2) corresponds to the Elrod–Adams model in air–fluid mixture region Ω_0. Eq. (2.3) is the condition for the Elrod–Adams flux on Σ. The boundary conditions that complete the model are given by Eqs. (2.4) and (2.5). To study the model problem, we introduce the space

$$V=\{\varphi\in H^1(\Omega)/\varphi|_\Gamma=0\}. \tag{2.7}$$

It has been proved that under certain hypotheses on the feed parameter θ_0 the solution q belongs to $H_0^1(\Omega)$ (see [19], for the details). However, in the present work this property is not needed and the existence of the solution proved in [19] by means of fixed-point techniques, allows to obtain that

$$\int_\Omega h^3\nabla q\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y=\int_\Omega \theta h\frac{\partial\varphi}{\partial x}\,\mathrm{d}x\,\mathrm{d}y+\int_{\Gamma_0}\theta_0 h\varphi\,\mathrm{d}\sigma,\quad \forall\varphi\in V, \tag{2.8}$$

$$\theta\in H(q)\quad\text{in }\Omega, \tag{2.9}$$

$$q\geqslant 0\quad\text{in }\Omega, \tag{2.10}$$

has a solution $q\in V$ and $\theta\in L^\infty(\Omega)$ for the given data $\theta_0\in L^2(\Gamma_0)$ such that $0\leqslant\theta_0\leqslant\theta_{\max}\leqslant 1$ where $\theta_{\max}$ is given by an auxiliary problem (see [19], for the details). In (2.9) H represents the Heaviside operator.

Moreover, the mathematical tools developed in the following sections can be extended to a more general gap function h provided that it verifies the hypotheses:

$$h=h(x,y),\quad h\in C^1(\overline{\Omega}),\quad h(x)\geqslant h_m>0,\quad h_m=\text{constant},$$

$$\partial h/\partial x\geqslant 0 \text{ in a neighborhood of } x=2\pi.$$

3. Discretization by means of characteristics method

In this section we propose a semidiscretization method for the previous free boundary problem. For this purpose, we consider that the left-hand side in Eq. (2.8) is a diffusion-type term while the right-hand side consists of the sum of a nonlinear convection term and a boundary term.

Among the different possibilities for the numerical discretization for linear convection terms in evolutive problems, the upwinding schemes are recommended in the literature to avoid oscillations. The convergence of its adaptation for the stationary linear convection-diffusion equation has been stated in [7] for the case of Dirichlet boundary conditions and under certain hypotheses for the velocity field. In this work, an error estimate is given when the unknown function is approximated by triangular finite elements of degree one and the velocity field is constant in each triangle. The pure convection problem with Dirichlet data is also analyzed.

For nonlinear problems, a general result of convergence does not seem to be found in the literature. Nevertheless, the method has been firstly used in [8] to approximate a periodic problem which modelizes the hydrodynamic lubrication of a circumferentially supplied journal bearing. Due to the periodic boundary conditions and the particular velocity field, the natural time discretization leads to a well-posed problem and a convergence theorem appears in a later work [9] for a noncoercive–related problem.

In this work, the main difficulty arises from the boundary conditions which are no longer periodic, preventing us from following the same way as in [7 or 8]. We have defined a modified semidiscretized scheme inducing a well-posed nonlinear problem for which we are able to give a result of convergence.

The departure point of the numerical method of characteristics applied to stationary problems consists of the introduction of an artificial dependence on time t in all the stationary functions. That is

$$\overline{\varphi}(x,y,t)=\varphi(x,y), \quad \overline{h}(x,y,t)=h(x,y),$$
$$\overline{\theta}(x,y,t)=\theta(x,y), \quad \overline{q}(x,y,t)=q(x,y).$$

Next step is the consideration of the artificial velocity field

$$\boldsymbol{u}(x,y)=(-1,0)$$

which allows to write Eq. (2.8) as an artificial evolutive one

$$\int_\Omega \overline{h}\,\overline{\theta}\frac{D\overline{\varphi}}{Dt}\,\mathrm{d}x\,\mathrm{d}y+\int_\Omega \overline{h}^3\,\nabla\overline{q}\nabla\overline{\varphi}\,\mathrm{d}x\,\mathrm{d}y=\int_{\Gamma_0}\theta_0\overline{h}\overline{\varphi}\,\mathrm{d}\sigma \tag{3.1}$$

with the total derivative given by

$$\frac{D\overline{\varphi}}{Dt}=\frac{\partial\overline{\varphi}}{\partial t}+\boldsymbol{u}\cdot\nabla\overline{\varphi}=-\frac{\partial\varphi}{\partial x}.$$

Because of the artificial dependence on time we consider the upwind approximation of the total derivative by

$$\frac{D\overline{\varphi}}{Dt}(x,y,t)\approx\frac{\varphi(x,y)-\varphi(X^k(x,y))}{k}, \tag{3.2}$$

where k is an artificial time step and $X^k(x,y)$ denotes the position at time $t-k$ of a particle placed in the point (x,y) at time t and is moving along the integral path of the velocity field $\boldsymbol{u}$. For this particular velocity field it is easy to conclude that

$$X^k(x,y)=(x+k,y)=X((x,y),t;t-k), \tag{3.3}$$

where in the general framework of the method, the function X is the solution of the final value problem associated with the differential equation of characteristics

$$\frac{\mathrm{d}}{\mathrm{d}\tau}X((x,y),t;\tau)=\boldsymbol{u}(X((x,y),t;\tau),\tau), \tag{3.4}$$

$$X((x,y),t;t)=(x,y). \tag{3.5}$$

In this way, taking into account the approximation (3.2) and the artificial dependence on time, Eq. (3.1) is formally approached by a family of equations depending on the parameter k

$$\int_\Omega h\theta\left(\frac{\varphi-\varphi\circ X^k}{k}\right)\mathrm{d}x\,\mathrm{d}y+\int_\Omega h^3\nabla q\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y=\int_{\Gamma_0}\theta_0 h\varphi\,\mathrm{d}\sigma \tag{3.6}$$

or, equivalently,

$$k\int_\Omega h^3\nabla q\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y+\int_\Omega h\theta\varphi\,\mathrm{d}x\,\mathrm{d}y=\int_\Omega h\theta(\varphi\circ X^k)\,\mathrm{d}x\,\mathrm{d}y+k\int_{\Gamma_0}\theta_0 h\varphi\,\mathrm{d}\sigma. \tag{3.7}$$

It appears that the integral

$$\int_\Omega h\theta(\varphi\circ X^k)\,\mathrm{d}x\,\mathrm{d}y \tag{3.8}$$

is not properly posed as $\varphi\circ X^k$ is not defined in Ω. After changing of variable and introducing the set

$$\Omega^k=(k,2\pi+k)\times(0,L),$$

we obtain that

$$\int_\Omega h\theta(\varphi\circ X^k)\,\mathrm{d}x\,\mathrm{d}y=\int_{\Omega^k}((h\theta)\circ X^{-k})\varphi\,\mathrm{d}x\,\mathrm{d}y \tag{3.9}$$

and the same difficulty about the definition of the test function φ in Ω^k still remains. The particles of the domain moved by characteristics must remain in the domain. In the periodic problem theoretically treated in [9] and the one numerically solved in [8], the periodic boundary conditions of the problem are essential to define function φ properly. The consideration of a velocity field tangential to the boundary (see [7, 8]) led to the same result. In the present situation due to the particular boundary conditions, in a first step, an appropriate function $\overline{Y}^{-k}$ could be defined by truncation of X^{-k} as

$$\overline{Y}^{-k}(x,y)=(\max\{x-k,0\},y),$$

whose graph is sketched in Fig. 2. Nevertheless, as there is no inverse function of $\overline{Y}^{-k}$, we propose to consider a sequence of functions $\{Y^{-k}\}\subset\mathscr{C}^1(\Omega)$ which approximate the graph of $\overline{Y}^{-k}$ (see Fig. 2) to which we will refer as the modified characteristics

$$Y^{-k}(x,y)=(Y_1^{-k}(x,y),Y_2^{-k}(x,y))=(x-k+\delta_k(x),y), \tag{3.10}$$

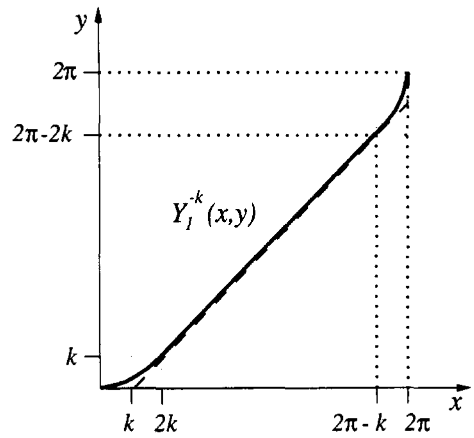

Fig. 2. Graph of the modified characteristics.

where $\delta_k(x)=0$ when $2k\leqslant x\leqslant 2\pi-k$, $-1\leqslant \delta_k'\leqslant 2$ and $\delta_k\rightarrow 0$ when $k\rightarrow 0$, so that the moved domain Ω^k remains as Ω with this modification. A possible choice of the function δ_k is

$$\delta_k(x)=\begin{cases}\frac{(x-2k)^2}{4k}, & x\in[0,2k],\\ 0, & x\in[2k,2\pi-k],\\ \frac{(x-2\pi+k)^2}{k}, & x\in[2\pi-k,2\pi].\end{cases} \tag{3.11}$$

So that, Eq. (3.7) is replaced by the modified equation

$$k\int_\Omega h^3\nabla q\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y+\int_\Omega h\theta\varphi\,\mathrm{d}x\,\mathrm{d}y=\int_\Omega (h\,\theta)\circ Y^{-k}|J^{-k}|\varphi\,\mathrm{d}x\,\mathrm{d}y+k\int_{\Gamma_0}\theta_0 h\varphi\,\mathrm{d}\sigma, \tag{3.12}$$

where $|J^{-k}|=1+\delta_k'(x)$ denotes the Jacobian of the transformation Y^{-k}.

The solution of the above formulation will be obtained by a fixed-point iteration method which can be thought as an explicit scheme for the artificial evolutive problem. So let us pose the problem:

For a given $\theta_k^n\in L^\infty(\Omega)$ such that $0\leqslant\theta_k^n\leqslant 1$

$$(\mathscr{P}_k^n)\begin{cases}(q_k^{n+1},\theta_k^{n+1})\in V\times L^\infty(\Omega),\\ k\int_\Omega h^3\nabla q_k^{n+1}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y+\int_\Omega h\theta_k^{n+1}\varphi\,\mathrm{d}x\,\mathrm{d}y\\ \quad=\int_\Omega (h\theta_k^n)\circ Y^{-k}|J^{-k}|\varphi\,\mathrm{d}x\,\mathrm{d}y+k\int_{\Gamma_0}\theta_0 h\varphi\,\mathrm{d}\sigma,\quad\forall\varphi\in V,\\ \theta_k^{n+1}\in H(q_k^{n+1}).\end{cases} \tag{3.13}$$

Proposition 3.1. *If θ_0 is nonnegative then any solution q_k^{n+1} of the problem $(\mathscr{P}_k^n)$ is nonnegative.*

Proof. The condition $\theta_k^{n+1} \in H(q_k^{n+1})$ implies that θ_k^{n+1} is nonnegative for all n. Then, by taking $\varphi = (q_k^{n+1})^-$ as test function in $(\mathscr{P}_k^n)$ where $(\cdot)^- = -\inf\{\cdot, 0\}$, the inequality

$$k \int_\Omega h^3 \nabla q_k^{n+1} \nabla (q_k^{n+1})^- \,\mathrm{d}x\,\mathrm{d}y \geqslant 0$$

holds or equivalently,

$$-k \int_\Omega h^3 |\nabla (q_k^{n+1})^-|^2 \,\mathrm{d}x\,\mathrm{d}y \geqslant 0.$$

Therefore,

$$|\nabla (q_k^{n+1})^-| = 0 \quad \text{in } \Omega \Rightarrow (q_k^{n+1})^- = c$$

and the boundary condition on Γ concludes the result. □

4. Existence, uniqueness and estimates for the solution of $(\mathscr{P}_k^n)$

In this paragraph the relation between the solution of $(\mathscr{P}_k^n)$ and the one of a variational inequality shall be studied. For this, let us define the problem

$$(\mathscr{I}\mathscr{V}2) \begin{cases} q_k^{n+1} \in V, \\ k \displaystyle\int_\Omega h^3 \nabla q_k^{n+1} \nabla (\varphi - q_k^{n+1}) \,\mathrm{d}x\,\mathrm{d}y + j(\varphi) - j(q_k^{n+1}) \\ \quad \geqslant \displaystyle\int_\Omega (h\,\theta_k^n) \circ Y^{-k}\, |J^{-k}|\, (\varphi - q_k^{n+1}) \,\mathrm{d}x\,\mathrm{d}y + k \int_{\Gamma_0} \theta_0 h (\varphi - q_k^{n+1}) \,\mathrm{d}\sigma, \quad \forall \varphi \in V, \end{cases} \tag{4.1}$$

where

$$j(\varphi) = \int_\Omega h \varphi^+ \,\mathrm{d}x\,\mathrm{d}y \tag{4.2}$$

with $\varphi^+ = \sup(\varphi, 0)$.

The above second kind variational inequality is related to the problem $(\mathscr{P}_k^n)$ by the following proposition.

Proposition 4.1. *If $(q_k^{n+1}, \theta_k^{n+1})$ is a solution of $(\mathscr{P}_k^n)$ then q_k^{n+1} is a solution of $(\mathscr{I}\mathscr{V}2)$.*

Proof. As the relation between θ_k^{n+1} and q_k^{n+1} is given by $\theta_k^{n+1} \in H(q_k^{n+1}) = \partial\psi(q_k^{n+1})$ with

$$\psi(\varphi) = \varphi^+, \tag{4.3}$$

we have

$$\varphi^+ - (q_k^{n+1})^+ \geqslant \theta_k^{n+1}(\varphi - q_k^{n+1}), \quad \text{a.e. in } \Omega \tag{4.4}$$

and the result is concluded from the definition of j and the formulation of $(\mathscr{P}_k^n)$. □

The above proposition states a first relation between the solution of both problems (($\mathscr{P}_k^n$) and ($\mathscr{IV}2$)). This is just a previous step to relate ($\mathscr{P}_k^n$) with the following first kind variational inequality ($\mathscr{IV}1$):

$$(\mathscr{IV}1)\begin{cases} q_k^{n+1}\in K, \\ k\displaystyle\int_\Omega h^3\nabla q_k^{n+1}\nabla(\varphi-q_k^{n+1})\,\mathrm{d}x\,\mathrm{d}y \geqslant \int_\Omega ((h\theta_k^n)\circ Y^{-k}|J^{-k}|-h)(\varphi-q_k^{n+1})\,\mathrm{d}x\,\mathrm{d}y \\ \quad +k\displaystyle\int_{\Gamma_0}\theta_0 h(\varphi-q_k^{n+1})\,\mathrm{d}\sigma, \quad \forall\varphi\in K \end{cases} \tag{4.5}$$

with

$$K=\{\varphi\in V/\varphi\geqslant 0 \text{ in } \Omega\},$$

in order to obtain the existence and uniqueness of solution, and a relation with the solution of the problem ($\mathscr{P}_k^n$).

Proposition 4.2. *The problem ($\mathscr{IV}1$) admits one and only one solution. Moreover, the unique solution q_k^{n+1} belongs to $\mathscr{C}^{0,\lambda}(\overline{\Omega})$ for $0<\lambda<1$ and verifies that*

$$0\leqslant q_k^{n+1}\leqslant C\left\{\frac{1}{k}\|(h\theta_k^n)\circ Y^{-k}|J^{-k}|-h\|_{L^2(\Omega)}+\|\theta_0 h\|_{L^2(\Gamma_0)}\right\} \quad \text{a.e. in } \Omega \tag{4.6}$$

for a constant C.

Proof. The existence and uniqueness of solution for ($\mathscr{IV}1$) and the estimate (4.6) are obtained by applying the classical results for variational inequalities appearing in [16, p. 140]. □

The equivalence between problems ($\mathscr{P}_k^n$) and ($\mathscr{IV}1$) may be proved as in [9] to obtain the following proposition:

Proposition 4.3. *If $(q_k^{n+1},\theta_k^{n+1})$ is a solution of ($\mathscr{P}_k^n$) then q_k^{n+1} is a solution of ($\mathscr{IV}1$). Moreover, if q_k^{n+1} is the solution of ($\mathscr{IV}1$) then there exists a unique θ_k^{n+1} such that the pair $(q_k^{n+1},\theta_k^{n+1})$ is a solution of ($\mathscr{P}_k^n$).*

Corollary 4.4. *There exists one and only one solution $(q_k^{n+1},\theta_k^{n+1})$ of problem ($\mathscr{P}_k^n$).*

5. Convergence of the semidiscretized problem

In the previous paragraph the existence and uniqueness of solution for each problem ($\mathscr{P}_k^n$) has been stated. In the present section the convergence of the sequence of solutions is analyzed. In order to do that, firstly some a priori estimates for q_k^{n+1} in V and θ_k^{n+1} in the dual space V^* of V are to be obtained. It must be noted that this kind of result is only present in the literature for the linear convection–diffusion problem in [7] and for a semicoercive problem with periodic boundary conditions in [9].

Proposition 5.1. *The solution of* $(\mathscr{P}_k^n)$ *verifies*

$$\|q_k^{n+1}\|_V \leqslant c_1, \tag{5.1}$$

$$\|h\theta_k^{n+1} - (h\theta_k^n)\circ Y^{-k}\,|J^{-k}|\|_{V^*} \leqslant kc_2 \tag{5.2}$$

with c_1 *and* c_2 *constants depending only on* Ω *and* h.

Proof. By taking the test functions $\varphi = 2q_k^{n+1}$ and $\varphi = 0$ in $(\mathscr{I}\mathscr{V}1)$ we obtain the identity

$$k\int_\Omega h^3|\nabla q_k^{n+1}|^2\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega ((h\theta_k^n)\circ Y^{-k}|J^{-k}| - h)q_k^{n+1}\,\mathrm{d}x\,\mathrm{d}y + k\int_{\Gamma_0}\theta_0 h q_k^{n+1}\,\mathrm{d}\sigma.$$

From the properties of h we have

$$\begin{aligned} h_m^3\|q_k^{n+1}\|_V^2 &\leqslant \int_\Omega \frac{((h\theta_k^n)\circ Y^{-k}|J^{-k}| - h)}{k}\,q_k^{n+1}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_0}\theta_0\,h\,q_k^{n+1}\,\mathrm{d}\sigma. \\ &\leqslant \int_\Omega \frac{(h\circ Y^{-k}|J^{-k}| - h)}{k}(\theta_k^n\circ Y^{-k})q_k^{n+1}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_0}\theta_0 h q_k^{n+1}\,\mathrm{d}\sigma, \end{aligned}$$

where the last step comes from the inequality

$$\theta_k^n\circ Y^{-k}\leqslant 1, q_k^{n+1}\geqslant 0 \quad\text{and}\quad h\geqslant 0 \Rightarrow h(\theta_k^n\circ Y^{-k})q_k^{n+1}\leqslant hq_k^{n+1}.$$

If we consider some one variable calculus computations available due to the regularity of h and δ_k, we have

$$\begin{aligned} \frac{h\circ Y^{-k}|J^{-k}| - h}{k}(x,y) &= \frac{h(x-k+\delta_k(x))(1+\delta_k'(x)) - h(x,y)}{k} \\ &= \frac{(h(x) + (k-\delta_k(x))h'(\xi))(1+\delta_k'(x)) - h(x)}{k}. \end{aligned}$$

Let us define

$$\begin{aligned} G(x,y) &= \left(\frac{(h\circ Y^{-k}|J^{-k}| - h)}{k}(\theta_k^n\circ Y^{-k})q_k^{n+1}\right)(x,y) \\ &= \frac{h(x-k+\delta_k(x))\,|J^{-k}| - h(x)}{k}\theta_k^n(Y^{-k}(x,y))q_k^{n+1}(x,y). \end{aligned}$$

So, we consider the decomposition of the integral

$$\int_\Omega G\,\mathrm{d}x\,\mathrm{d}y = \int_{(0,2k)\times(0,1)} G\,\mathrm{d}x\,\mathrm{d}y + \int_{(2k,2\pi-k)\times(0,1)} G\,\mathrm{d}x\,\mathrm{d}y + \int_{(2\pi-k,2\pi)\times(0,1)} G\,\mathrm{d}x\,\mathrm{d}y. \tag{5.3}$$

For technical reasons we assume that k is strictly less than one. Thus, from the properties of $\delta_k(x)$

$$0\leqslant\delta_k(x)\leqslant k \quad \text{for } 0\leqslant x\leqslant 2\pi, \tag{5.4}$$

$$-1\leqslant\delta_k'(x)\leqslant 0 \quad \text{for } 0\leqslant x\leqslant 2k, \tag{5.5}$$

$$\delta_k'(x) = 0 \quad \text{for } 2k \leqslant x \leqslant 2\pi - k, \tag{5.6}$$

$$0 \leqslant \delta_k'(x) \leqslant 2 \quad \text{for } 2\pi - k \leqslant x \leqslant 2\pi \tag{5.7}$$

and the introduction of the definitions

$$\overline{H} = \max_{(x,y)\in\overline{\Omega}} |h'(x)| = \varepsilon,$$

$$H = \max_{(x,y)\in\overline{\Omega}} |h(x)| = 1 + \varepsilon,$$

in terms of the function $h \in \mathscr{C}^1(\overline{\Omega})$, we have the bounds

$$\frac{1}{k}\int_{(0,2k)\times(0,1)} h\delta_k' (\theta_k^n \circ Y^{-k}) q_k^{n+1} \, dx\, dy \leqslant 0,$$

from (5.5) and the inequality $h\delta_k'(\theta_k^n \circ Y^{-k}) q_k^{n+1} \leqslant 0$ in the set $(0,2k)\times(0,1)$;

$$\frac{1}{k}\int_{(0,2k)\times(0,1)} (k - \delta_k) h'(\xi)(1 + \delta_k')(\theta_k^n \circ Y^{-k}) q_k^{n+1} \, dx\, dy \leqslant \overline{H}|\Omega|^{1/2} k_1(\Omega) \|q_k^{n+1}\|_V,$$

from (5.4) and (5.5), the definition of $\overline{H}$ and with $k_1(\Omega)$ the Poincaré constant;

$$\frac{1}{k}\int_{(2k,2\pi-k)\times(0,1)} (h\delta_k' + (k - \delta_k) h'(\xi)(1 + \delta_k'))(\theta_k^n \circ Y^{-k}) q_k^{n+1} \, dx\, dy \leqslant \overline{H}|\Omega|^{1/2} k_1(\Omega) \|q_k^{n+1}\|_V,$$

from (5.4) and (5.6) and the definition of $\overline{H}$;

$$\frac{1}{k}\int_{(2\pi-k,2\pi)\times(0,1)} (k - \delta_k) h'(\xi)(1 + \delta_k')(\theta_k^n \circ Y^{-k}) q_k^{n+1} \, dx\, dy \leqslant 3\overline{H}|\Omega|^{1/2} k_1(\Omega) \|q_k^{n+1}\|_V,$$

from (5.4) and (5.7) and the definition of $\overline{H}$. Moreover, we have the inequality

$$\frac{1}{k}\int_{(2\pi-k,2\pi)\times(0,1)} h\delta_k'(\theta_k^n \circ Y^{-k}) q_k^{n+1} \, dx\, dy \leqslant \frac{1}{k}\int_{(2\pi-k,2\pi)\times(0,1)} h\delta_k' q_k^{n+1} \, dx\, dy$$

and the use of the Green formula in the last integral leads to

$$\begin{aligned}
\frac{1}{k}\int_{(2\pi-k,2\pi)\times(0,1)} h\delta_k' q_k^{n+1} \, dx\, dy &= -\frac{1}{k}\int_{(2\pi-k,2\pi)\times(0,1)} h'\delta_k \, q_k^{n+1} \, dx\, dy \\
&\quad - \frac{1}{k}\int_{(2\pi-k,2\pi)\times(0,1)} h\delta_k \frac{\partial q_k^{n+1}}{\partial x} \, dx\, dy \\
&\quad + \frac{1}{k}\int_{\partial((2\pi-k,2\pi)\times(0,1))} h\delta_k \, q_k^{n+1} \boldsymbol{m}\cdot\boldsymbol{\imath} \, d\sigma \\
&\leqslant \frac{1}{k}\int_{(2\pi-k,2\pi)\times(0,1)} h\,\delta_k \left|\frac{\partial q_k^{n+1}}{\partial x}\right| dx\, dy,
\end{aligned}$$

where the last inequality is due to the fact that

$$\begin{aligned}
&0\leqslant\delta_k(x)\leqslant k \quad \text{for } 0\leqslant x\leqslant 2\pi,\\
&h'(x)\geqslant 0 \quad \text{for } \pi\leqslant 2\pi-k\leqslant x\leqslant 2\pi,\\
&q_k^{n+1}\geqslant 0 \quad \text{in } \Omega,\\
&q_k^{n+1}=0 \quad \text{on } \Gamma,\\
&\delta_k\, h\, q_k^{n+1}\, \boldsymbol{m}\cdot\boldsymbol{\imath}\leqslant 0 \quad \text{for } x=2\pi-k.
\end{aligned}$$

Therefore, we have

$$\frac{1}{k}\int_{(2\pi-k,2\pi)\times(0,1)} h\delta_k'(\theta_k^n\circ Y^{-k})q_k^{n+1}\,\mathrm{d}x\,\mathrm{d}y\leqslant H|\Omega|^{1/2}\|q_k^{n+1}\|_V$$

and

$$h_m^3\|q_k^{n+1}\|_V^2\leqslant(5\overline{H}\,k_1(\Omega)+H)|\Omega|^{1/2}\|q_k^{n+1}\|_V+H|\Gamma_0|^{1/2}k_2(\Omega)\|q_k^{n+1}\|_V.$$

Finally, we can state

$$\|q_k^{n+1}\|_V\leqslant\frac{((5\overline{H}\,k_1(\Omega)+H)|\Omega|^{1/2}+H|\Gamma_0|^{1/2}k_2(\Omega))}{h_m^3}=c_1(\Omega), \tag{5.8}$$

where $k_1(\Omega)$ and $k_2(\Omega)$ come from the Poincaré inequality and the norm of the trace mapping from V into $L^2(\Gamma_0)$.

In order to obtain (5.2) we consider the identity (3.13)

$$\int_\Omega((h\theta_k^{n+1})-(h\theta_k^n)\circ Y^{-k}|J^{-k}|)\varphi\,\mathrm{d}x\,\mathrm{d}y=-k\int_\Omega h^3\nabla q_k^{n+1}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y+k\int_{\Gamma_0}\theta_0 h\varphi\,\mathrm{d}\sigma,\quad\forall\varphi\in V$$

and therefore

$$\begin{aligned}
\left|\int_\Omega((h\theta_k^{n+1})-(h\theta_k^n)\circ Y^{-k}|J^{-k}|)\varphi\,\mathrm{d}x\,\mathrm{d}y\right| &\leqslant kH^3\|q_k^{n+1}\|_V\|\varphi\|_V+kH|\Gamma_0|^{1/2}\|\varphi\|_{L^2(\Gamma_0)}\\
&\leqslant (kH^3c_1(\Omega)+kH|\Gamma_0|^{1/2}k_2(\Omega))\|\varphi\|_V
\end{aligned}$$

which concludes the estimate (5.2) for c_2 given by

$$c_2=H^3c_1(\Omega)+H|\Gamma_0|^{1/2}k_2(\Omega).\qquad\square \tag{5.9}$$

Proposition 5.2. *The sequence q_k^{n+1} converges to an element q in V and the sequence θ_k^{n+1} tends to θ in $L^\infty(\Omega)$ when $k\to 0$ and $n\to\infty$. Moreover, the pair (q,θ) is a solution of the weak formulation (2.8)–(2.10).*

Proof. In the identity

$$k\int_\Omega h^3\nabla q_k^{n+1}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y+\int_\Omega(h\theta_k^{n+1}-(h\theta_k^n)\circ Y^{-k}|J^{-k}|)\varphi\,\mathrm{d}x\,\mathrm{d}y=k\int_{\Gamma_0}\theta_0 h\varphi\,\mathrm{d}\sigma,$$

we consider the change of variable defined by Y^k, the inverse of Y^{-k}, in the integral

$$\int_\Omega (h\theta_k^n)\circ Y^{-k}|J^{-k}|\varphi\,\mathrm{d}x\,\mathrm{d}y=\int_\Omega h\theta_k^n(\varphi\circ Y^k)\,\mathrm{d}x\,\mathrm{d}y$$

which leads to the identity

$$\int_\Omega h^3\nabla q_k^{n+1}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega h\theta_k^n\frac{(\varphi\circ Y^k)-\varphi}{k}\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int_{\Gamma_0}\theta_0 h\varphi\,\mathrm{d}\sigma-\frac{1}{k}\int_\Omega h(\theta_k^{n+1}-\theta_k^n)\varphi\,\mathrm{d}x\,\mathrm{d}y,\quad \forall\varphi\in\mathscr{D}(\Omega). \tag{5.10}$$

Moreover, from the estimates of the previous proposition we can deduce by means of compacity arguments that

$$\exists q \text{ such that}/q_k^{n+1}\to q \text{ weak in } V, \tag{5.11}$$

$$\exists\theta \text{ such that}/\theta_k^{n+1}\to\theta \text{ weak-}* \text{ in } L^\infty(\Omega), \tag{5.12}$$

where both limits are taken when $k\to 0$ and $n\to\infty$. On the other hand, the choice of the modified characteristics (3.11) and the fact that $|J^k|$, the Jacobian of the transformation Y^k, is uniformly bounded implies that

$$\int_\Omega h\theta_k^n\frac{(\varphi\circ Y^k)-\varphi}{k}\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int_{(2k,2\pi-k)\times(0,L)}\left(h\theta_k^n\frac{\partial\varphi}{\partial x}+0(k)\right)\mathrm{d}x\,\mathrm{d}y+\int_{((0,2k)\cup(2\pi-k))\times(0,L)}h\theta_k^n\frac{(\varphi\circ Y^k)-\varphi}{k}\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\int_\Omega h\theta_k^n\frac{(\varphi\circ Y^k)-\varphi}{k}\,\mathrm{d}x\,\mathrm{d}y\to\int_{(0,2\pi)\times(0,L)}h\theta\frac{\partial\varphi}{\partial x}\,\mathrm{d}x\,\mathrm{d}y,\quad\forall\varphi\in\mathscr{D}(\Omega). \tag{5.13}$$

Then, (5.11) allows to conclude from (5.10) the equation

$$\int_\Omega h^3\nabla q\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega h\theta\frac{\partial\varphi}{\partial x}\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int_{\Gamma_0}\theta_0 h\varphi\,\mathrm{d}\sigma-\lim_{n\to\infty}\lim_{k\to 0}\frac{1}{k}\int_\Omega h(\theta_k^{n+1}-\theta_k^n)\varphi\,\mathrm{d}x\,\mathrm{d}y,\quad\forall\varphi\in\mathscr{D}(\Omega). \tag{5.14}$$

It remains to prove that

$$\lim_{n\to\infty}\lim_{k\to 0}\frac{1}{k}\int_\Omega h(\theta_k^{n+1}-\theta_k^n)\varphi\,\mathrm{d}x\,\mathrm{d}y=0,\quad\forall\varphi\in\mathscr{D}(\Omega).$$

For this purpose we consider that

$$\frac{1}{k}\int_\Omega h(\theta_k^{n+1}-\theta_k^n)\varphi\,\mathrm{d}x\,\mathrm{d}y$$
$$=\frac{1}{k}\int_\Omega (h\theta_k^{n+1}-(h\theta_k^n)\circ Y^{-k}|J^{-k}|)\varphi\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{k}\int_\Omega((h\theta_k^n)\circ Y^{-k}|J^{-k}|-h\theta_k^n)\varphi\,\mathrm{d}x\,\mathrm{d}y$$
$$=\frac{1}{k}\int_\Omega (h\theta_k^{n+1}-(h\theta_k^n)\circ Y^{-k}|J^{-k}|)\varphi\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{k}\int_\Omega h\theta_k^n(\varphi\circ Y^k-\varphi)\,\mathrm{d}x\,\mathrm{d}y,\quad \forall\varphi\in\mathscr{D}(\Omega)$$

and therefore taking into account the estimate (5.2) and using the same technique as in (5.13) we can state that

$$\left\|\frac{\theta_k^{n+1}-\theta_k^n}{k}\right\|_{V^*},\leqslant c_3,$$

where c_3 is a constant which does not depend on n and k due to the regularity of φ and provided we choose the parameter k lower than one, for example.

If we define

$$\overline{\theta}_k^n(x,y,t)=\frac{\theta_k^{n+1}(x,y)-\theta_k^n(x,y)}{k}(t-nk)+\theta_k^n(x,y),\quad \forall t\in[nk,(n+1)k], \tag{5.15}$$

then

$$\|\overline{\theta}_k^n\|_{V^*}\leqslant c_3k+|\Omega|,\quad \forall t\geqslant 0. \tag{5.16}$$

Thus, if for any $T>0$ we choose a subsequence such that $k(n+1)\leqslant T$ then

$$\|\overline{\theta}_k^n\|_{H^1(0,T;V^*)}\leqslant c_4. \tag{5.17}$$

The previous inequality implies that there exists $\overline{\theta}\in H^1(0,T;V^*)$ such that the following weak convergences in $L^2(0,T;V^*)$ hold:

$$\overline{\theta}_k^n\to\overline{\theta},$$
$$\frac{\partial\overline{\theta}_k^n}{\partial t}\to\frac{\partial\overline{\theta}}{\partial t},$$

but from the definition (5.15) we have

$$\|\overline{\theta}_k^n-\theta_k^n\|_{V^*}\leqslant\|\theta_k^{n+1}-\theta_k^n\|_{V^*}\leqslant kc_3,\quad \forall t\geqslant 0 \tag{5.18}$$

and therefore, taking the limit in n and k, we have $\overline{\theta}=\theta$ with

$$\frac{\partial\theta}{\partial t}=0. \tag{5.19}$$

On the other hand, by taking derivatives in (5.15) we have

$$\frac{\theta_k^{n+1}-\theta_k^n}{k}=\frac{\partial\overline{\theta}_k^n}{\partial t}\to\frac{\partial\theta}{\partial t}=0,$$

so that, passing to the limit in (5.14), we can state the proof. □

6. Algorithm and numerical results

The numerical algorithm is based on the variational inequality formulation ($\mathscr{I}\mathscr{V}1$) of the semidiscretized problem. Thus, its numerical solution can be placed in the frame of duality methods for variational inequalities proposed in [6]. In this work, the authors use some classical results for monotone operators which can be found in [10], for example.

Let G be a maximal monotone operator and let ω and λ be two nonnegative parameters verifying the condition $\lambda\omega < 1$, it can be proved that the operator

$$J_\lambda^\omega = ((1-\lambda\omega)I + \lambda G)^{-1}$$

is well defined as the resolvent operator of G. Moreover, the Yosida approximation of $G - \omega I$ is defined as

$$G_\lambda^\omega = \frac{I - J_\lambda^\omega}{\lambda}$$

and G_λ^ω is lipschitcian with constant λ^{-1}.

The justification of the proposed algorithm is based on the following lemma:

Lemma 6.1. *Let G be a maximal monotone operator in a Hilbert space V then the following conditions are equivalent*:
1. $u \in G(y) - \omega y$,
2. $u = G_\lambda^\omega(y + \lambda u)$.

Therefore, in the problem ($\mathscr{P}_k^n$) we consider that the Heaviside operator H is the subdifferential of the function

$$\psi(z) = \max(z, 0)$$

and then it is maximal monotone. In order to apply the previous lemma it is useful to introduce the new unknown r^{n+1} defined by

$$r_k^{n+1} \in H(q_k^{n+1}) - \omega q_k^{n+1}$$

and to write the problem ($\mathscr{P}_k^n$) as

$$(\mathscr{P}_k^n)\begin{cases} (q_k^{n+1}, r_k^{n+1}) \in V \times L^\infty(\Omega), \\ k\displaystyle\int_\Omega h^3\, \nabla q_k^{n+1}\, \nabla\varphi \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega h\,(\omega q_k^{n+1} + r_k^{n+1})\,\varphi\,\mathrm{d}x\,\mathrm{d}y \\ \quad = \displaystyle\int_\Omega (h\theta_k^n)\circ Y^{-k}|J^{-k}|\varphi\,\mathrm{d}x\,\mathrm{d}y + k\int_{\Gamma_0} \theta_0 h\varphi\,\mathrm{d}\sigma, \quad \forall\varphi \in V, \\ r_k^{n+1} \in H(q_k^{n+1}) - \omega q_k^{n+1} \end{cases} \tag{6.1}$$

and therefore, from Lemma 6.1, we have

$$r_k^{n+1} = H_\lambda^\omega(q_k^{n+1} + \lambda r_k^{n+1}), \tag{6.2}$$

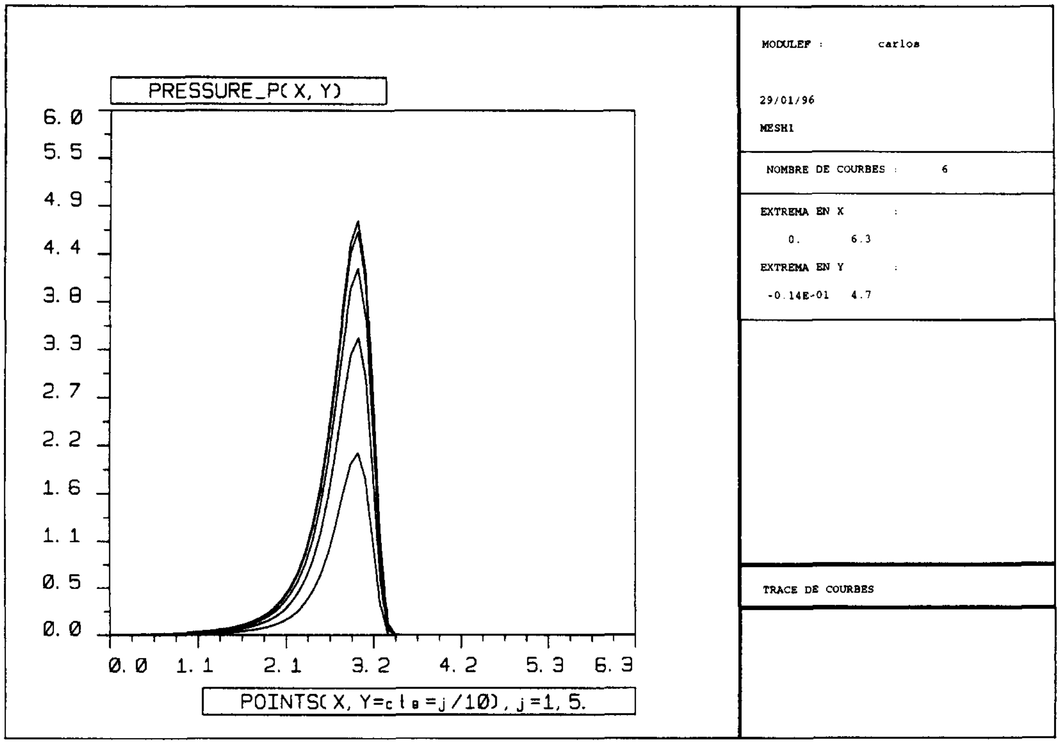

Fig. 3. Cross section of pressure for various y for MESH1 ($k=l$).

where H_λ^ω represents the Yosida approximation of $H-\omega I$. Thus, the final algorithm is based on the updating of the term r_k^{n+1} in Eq. (6.2) and the numerical solution of the linear problem in each step. The scheme can be sketched in the following way:

- $n=0$.
- Loop in n (characteristics).
- $j=0$.
- Loop in j (updating of r_k^{n+1}).
  – Numerical solution of

$$(\mathscr{P}_k^{n,j})\begin{cases} q_k^{n+1,j}\in V, \\ k\displaystyle\int_\Omega h^3\nabla q_k^{n+1,j}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y+\omega\int_\Omega hq_k^{n+1,j}\varphi\,\mathrm{d}x\,\mathrm{d}y \\ \quad =\displaystyle\int_\Omega(h\theta_k^n)\circ Y^{-k}\,|J^{-k}|\varphi\,\mathrm{d}x\,\mathrm{d}y+k\int_{\Gamma_0}\theta_0h\varphi\,\mathrm{d}\sigma \\ \quad -\displaystyle\int_\Omega hr_k^{n+1,j}\varphi\,\mathrm{d}x\,\mathrm{d}y,\quad\forall\varphi\in V. \end{cases}\tag{6.3}$$

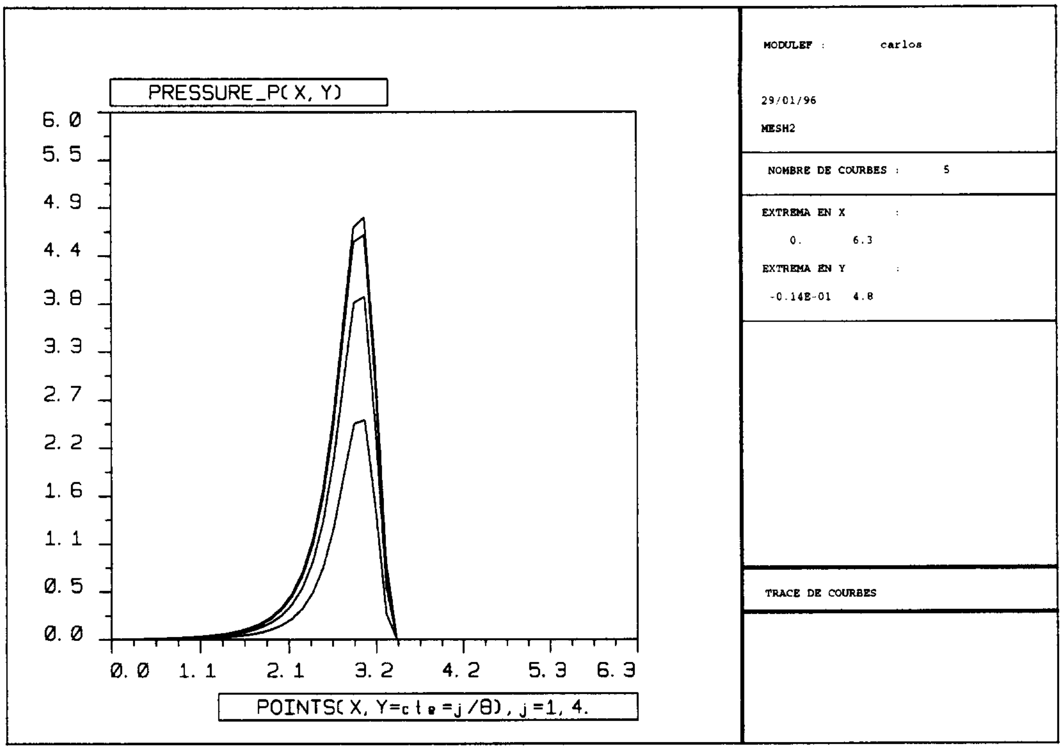

Fig. 4. Cross sections of pressure for various y for MESH2 ($k=l$).

– Updating of $r_k^{n+1,j}$

$$r_k^{n+1,j+1} = H_\lambda^\omega(q_k^{n+1,j} + \lambda r_k^{n+1,j}). \tag{6.4}$$

- Convergence in j.
- Convergence in n.

The theoretical convergence of $q_k^{n+1,j}$ to q_k^{n+1} and of $r_k^{n+1,j}$ to r_k^{n+1} has been analyzed in [6] in the general framework of the numerical approximation of variational inequalities and the convergence of q_k^{n+1} to q and θ_k^{n+1} to θ has been studied in the previous section.

For the numerical solution of the linear problem $(\mathscr{P}_k^{n,j})$ we propose the Lagrange finite element of degree one. Thus, for a strictly positive number l and a triangulation $\{\tau_l\}$ of the domain Ω, we introduce the classical notation (see [15], for example):

$$V_l = \{\varphi_l \in C^0(\overline{\Omega}) / \varphi_l|_U \in P_1, \forall U \in \tau_l\},$$

$$K_{0l} = \{\varphi_l \in V_l / \varphi_l|_\Gamma = 0\}$$

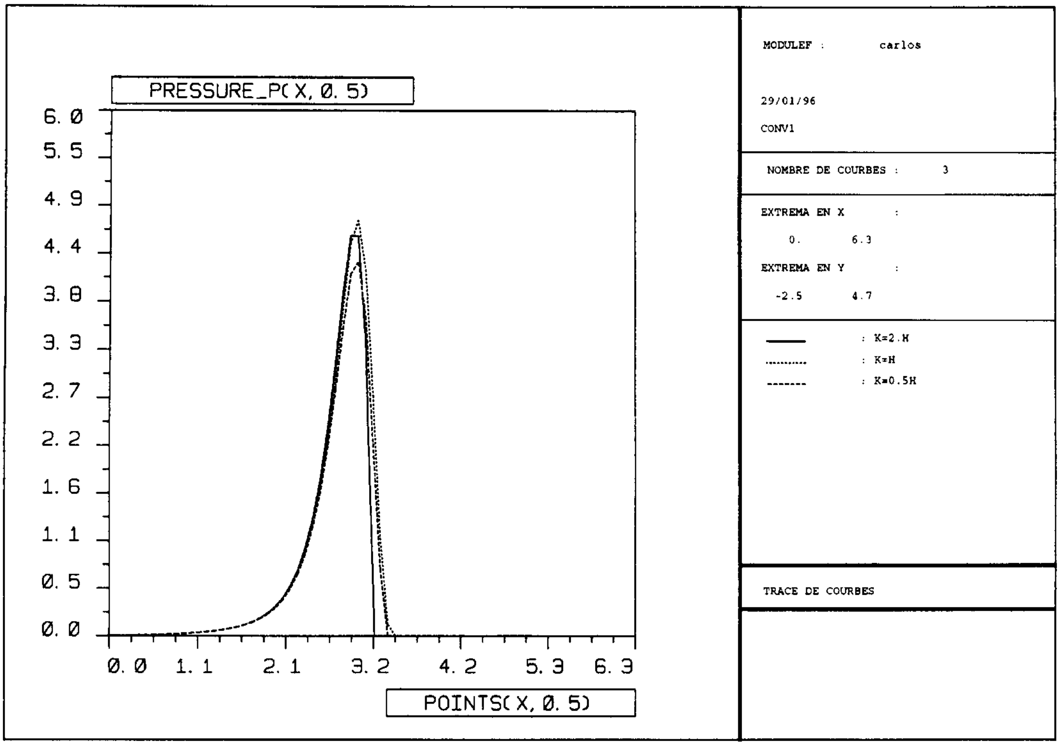

Fig. 5. Pressure at $(x, 0.5)$ for MESH1 $(k/l = 0.5, 1., 2.)$.

and we pose the discretized problem

$$(\mathscr{P}_{k,l}^{n,j})\begin{cases} q_{k,l}^{n+1,j} \in K_{0l}, \\ k\displaystyle\int_\Omega h^3 \nabla q_{k,l}^{n+1,j} \nabla \varphi_l \,\mathrm{d}x\,\mathrm{d}y + \omega \int_\Omega h q_{k,l}^{n+1,j} \varphi_l \,\mathrm{d}x\,\mathrm{d}y \\ \quad = \displaystyle\int_\Omega (h\theta_k^n) \circ Y^{-k} |J^{-k}| \varphi_l \,\mathrm{d}x\,\mathrm{d}y + k \int_{\Gamma_0} \theta_0 h \varphi_l \,\mathrm{d}\sigma \\ \quad - \displaystyle\int_\Omega h r_{k,l}^{n+1,j} \varphi_l \,\mathrm{d}x\,\mathrm{d}y, \quad \forall \varphi_l \in K_{0l}. \end{cases} \tag{6.5}$$

We state the following proposition whose proof is trivial.

Proposition 6.1. *The problem* $(\mathscr{P}_{k,l}^{n,j})$ *admits a unique solution.*

The previous discretization allows us to introduce the set of triangulation nodes

$$N_l = \{N \in \overline{\Omega} / N \text{ vertex of } \tau_l\}$$

and the functions φ_l^i defined by $\varphi_l^i(N_j) = \delta_{ij}$ for all N_j in N, which form a basis of V_l. These standard tools lead to the linear system

$$(kD + \omega M)Q = f, \tag{6.6}$$

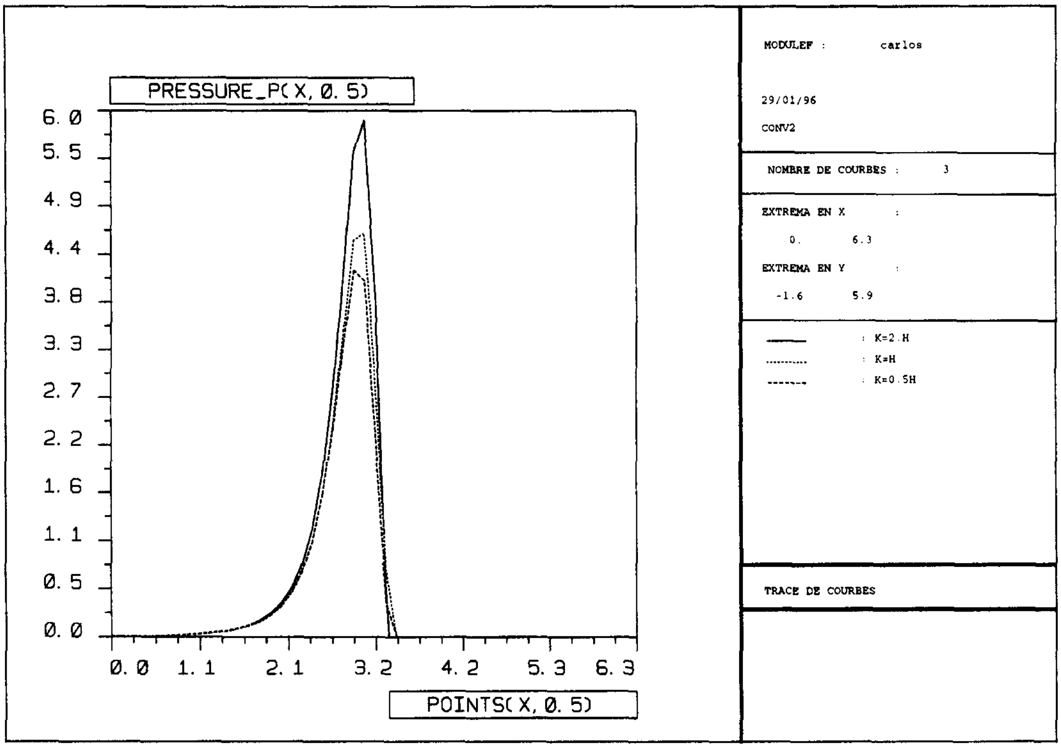

Fig. 6. Pressure at $(x, 0.5)$ for MESH2 $(k/l = 0.5, 1., 2.)$.

where D is the stiffness matrix, M the mass matrix, f the second member vector, and Q the unknown vector of nodes pressure.

The formulae for the coefficients, the building of the matrices and the details about the computer implementation are the classical ones in finite elements. Nevertheless, it must be pointed out that for the discretization of the term $r_{k,l}^{n+1,j}$, which is updated at each mesh node, we have chosen the trapezoidal rule for numerical integration which corresponds to approximate $r_{k,l}^{n+1,j}$ by the same finite element space that $q_{k,l}^{n+1,j}$. In several test examples the constant by element approximation of $r_{k,l}^{n+1,j}$, that is (P_0), has been carried out and the presence of oscillations in the final concentration has been observed.

The numerical results presented here have been obtained for the following set of physical data:

$$\varepsilon = 0.9, \quad \theta_0 = 0.9.$$

The finite element discretization has been performed with several meshes and we present the results corresponding to two of them: MESH1 and MESH2. The finer grid is MESH1 and contains 71×11 nodes (so 1400 elements and 721 nodes) which implies that $l = 0.089759771$ and MESH2 contains 51×9 nodes (so 800 elements and 459 nodes) which implies that $l = 0.125664$. Figs. 3 and 4 show the cuts of pressure with the choice $k = l$ for both grids. Due to the symmetry of the

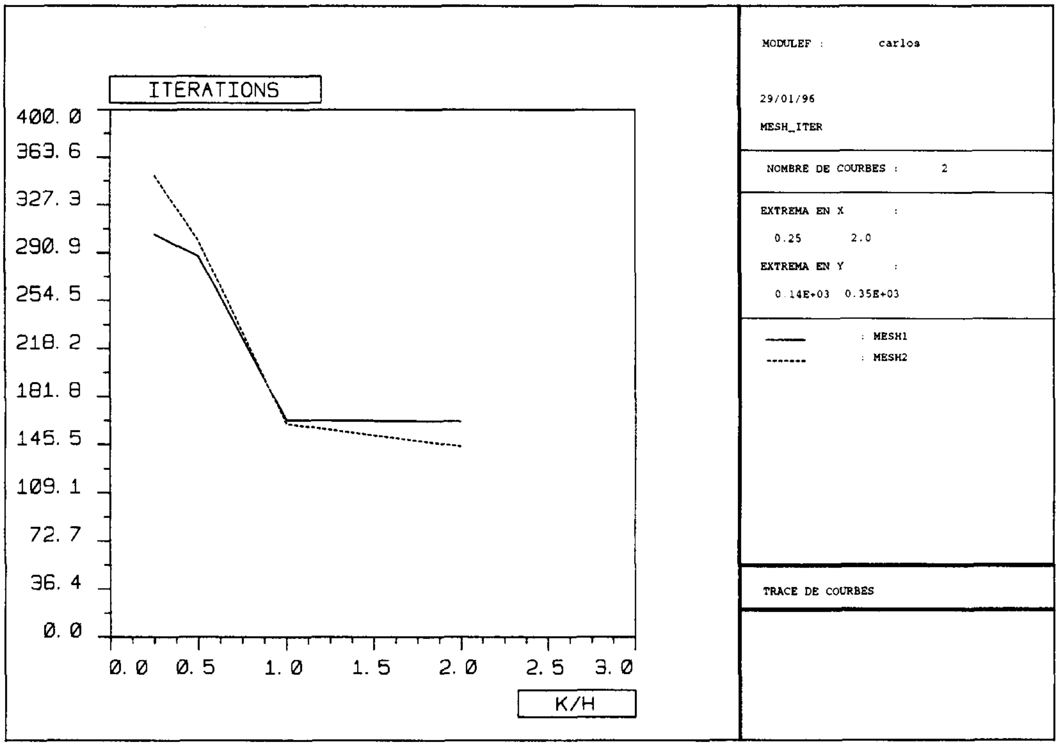

Fig. 7. Number of iterations.

Table 1
Maxima of the pressure for different k–l pairs

P_{max}	$l=2\pi/70$	$l=2\pi/50$
$k=2l$	4.57	5.89
$k=l$	4.74	4.80
$k=l/2$	4.27	4.18
$k=l/4$	4.05	3.93

pressure with respect to $y=0.5$ only six cuts are presented for MESH1 (five cuts for MESH2). Thus, when y tends to 0.5 the cuts show that the values of pressure increase.

In order to point out the convergence of the algorithm, Figs. 5 and 6 show the cuts of pressure in the line $y=0.5$ with different ratios k/l for MESH1 and MESH2, respectively. Finally Fig. 7 concludes an increasing number of iteration for decreasing values of k/l.

In Table 1 the maxima of the pressure distributions for different meshes and several values of the parameter k are presented. Following Fig. 7, a choice $k \leqslant l$ seems necessary. Due to the increasing number of iterations when k/l tends to zero and the facility of practical use, the best choice is $k=l$.

The numerical results here presented have been obtained for the error tests:

$$\frac{|\theta_k^{n+1} - \theta_k^n|_\infty}{|\theta_k^{n+1}|_\infty} \leqslant 10^{-4},$$

$$\frac{|q_k^{n+1} - q_k^n|_\infty}{|q_k^{n+1}|_\infty} \leqslant 10^{-4}.$$

7. Conclusions

We are able to introduce a well-posed full space time discretized problem which is nonlinear due to the concentration term. The result of convergence for the scheme of characteristics applied to a particular lubrication–cavitation model justifies its application in order to approximate the particular (convection type) nonlinearity involved. Numerical tests have been performed by using the Bermúdez–Moreno algorithm combined with Lagrange finite elements of degree one to solve the full-discretized nonlinear problem. A theoretical analysis of the convergence of this problem to the continuous one is an open problem. Another possible application of this work could be the numerical solution of some dam problems as, for example, the one posed in [13].

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